FOUR-BODY BOUND STATE FORMULATION IN THREE-DIMENSIONAL APPROACH (WITHOUT ANGULAR MOMENTUM DECOMPOSITION)

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The four-body bound state with two-body forces is formulated by the Three-Dimensional approach, which greatly simplifies the numerical calculations of few-body systems without performing the Partial Wave components. We have obtained the Yakubovsky equations directly as three dimensional integral equations.

1. Introduction

The four-body bound state calculations are traditionally carried out by solving coupled Yakubovsky equations in a partial wave basis. After truncation this leads to two coupled sets of finite number of coupled equations in three variables for the amplitudes. This is performed in configuration space[1] and in momentum space[2,3]. Though a few partial waves often provide qualitative insight, modern four-body calculations need 1572 or more different spin, isospin and angular momentum combinations[3]. It appears therefore natural to avoid a partial wave representation completely and work directly with vector variables. This is a common practice in four-body bound state calculations based on other techniques[4-9]. In recent years W. Glöckle and collaborators have introduced the three-dimensional approach which greatly simplifies the two- and three-body scattering and bound state calculations without using partial wave decomposition[10-12]. In this paper we extend this approach for four-body bound state with two-body interactions, we work directly with momentum vector variables in the Yakubovsky scheme. As a simplification we neglect spin and isospin degrees of freedom and treat four-boson bound state. Although the four-boson bound state has been studied with short-range forces and large scattering length at leading order in an effective quantum mechanics approach[13], but it is also based on partial wave approach.

2. Momentum Space Representation of Yakubovsky Equations in 3-D approach

The bound state of four identical bosons which interact via pairwise forces is given by coupled Yakubovsky equations[13]:

$$
\begin{aligned}
|\psi_1\rangle &= G_0 t_{12} P[(1+P_{34})|\psi_1\rangle + |\psi_2\rangle] \\
|\psi_2\rangle &= G_0 t_{12} \tilde{P}[(1+P_{34})|\psi_1\rangle + |\psi_2\rangle]
\end{aligned} \tag{1}
$$

In order to solve coupled equations, $Eq.(1)$, in momentum space we introduce the four-body basis states corresponding to each Yakubovsky component:

$$
\begin{aligned}
&|\vec{u}_1\,\vec{u}_2\,\vec{u}_3\rangle \\
&|\vec{v}_1\,\vec{v}_2\,\vec{v}_3\rangle
\end{aligned} \tag{2}
$$

Let us now represent coupled equations, $Eq.(1)$, with respect to the basis states have been introduced in Eq.(2):

$$
\begin{aligned}
\langle \vec{u}_1\,\vec{u}_2\,\vec{u}_3|\psi_1\rangle &= \int D^3u^{'} \int D^3u^{"}\ \langle \vec{u}_1\,\vec{u}_2\,\vec{u}_3|G_0tP|\vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}\rangle \\
&\qquad \langle \vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}|(1+P_{34})|\vec{u}_1^{"}\,\vec{u}_2^{"}\,\vec{u}_3^{"}\rangle\ \langle \vec{u}_1^{"}\,\vec{u}_2^{"}\,\vec{u}_3^{"}|\psi_1\rangle \\
&+ \int D^3u^{'} \int D^3v^{'}\ \langle \vec{u}_1\,\vec{u}_2\,\vec{u}_3|G_0tP|\vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}\rangle \\
&\qquad \langle \vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}|\vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}\rangle\ \langle \vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}|\psi_2\rangle \\
\langle \vec{v}_1\,\vec{v}_2\,\vec{v}_3|\psi_2\rangle &= \int D^3v^{'} \int D^3u^{'}\ \langle \vec{v}_1\,\vec{v}_2\,\vec{v}_3|G_0t\tilde{P}|\vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}\rangle \\
&\qquad \langle \vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}|(1+P_{34})|\vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}\rangle \langle \vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}|\psi_1\rangle \\
&+ \int D^3v^{'}\ \langle \vec{v}_1\,\vec{v}_2\,\vec{v}_3|G_0t\tilde{P}|\vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}\rangle \vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}\rangle \langle \vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}|\psi_2\rangle
\end{aligned} \tag{3}
$$

Where $D^3A = d^3A_1\, d^3A_2\, d^3A_3$. After evaluating the following matrix elements:

$$
\begin{aligned}
\langle \vec{u}_1\,\vec{u}_2\,\vec{u}_3|G_0tP|\vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}\rangle &= \frac{\delta^3(\vec{u}_3 - \vec{u}_3^{'})}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \\
&\quad \{\ \delta^3(\vec{u}_2 + \vec{u}_1^{'} + \frac{1}{2}\vec{u}_2^{'})\ \langle \vec{u}_1|t(\epsilon)|\frac{1}{2}\vec{u}_2 + \vec{u}_2^{'}\rangle \\
&\quad +\delta^3(\vec{u}_2 - \vec{u}_1^{'} + \frac{1}{2}\vec{u}_2^{'})\ \langle \vec{u}_1|t(\epsilon)|\frac{-1}{2}\vec{u}_2 - \vec{u}_2^{'}\rangle \} \\
\langle \vec{v}_1\,\vec{v}_2\,\vec{v}_3|G_0t\tilde{P}|\vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}\rangle &= \frac{\delta^3(\vec{v}_2 + \vec{v}_2^{'})\,\delta^3(\vec{v}_3 - \vec{v}_1^{'})}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}}\ \langle \vec{v}_1|t(\epsilon^*)|\vec{v}_3^{'}\rangle \\
\langle \vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}|(1+P_{34})|\vec{u}_1^{"}\,\vec{u}_2^{"}\,\vec{u}_3^{"}\rangle &= \delta^3(\vec{u}_1^{'} - \vec{u}_1^{"}) \\
&\times \{\ \delta^3(\vec{u}_2^{'} - \vec{u}_2^{"})\,\delta^3(\vec{u}_3^{'} - \vec{u}_3^{"}) \\
&\quad +\delta^3(\vec{u}_2^{'} - \frac{1}{3}\vec{u}_2^{"} - \frac{8}{9}\vec{u}_3^{"})\,\delta^3(\vec{u}_3^{'} - \vec{u}_2^{"} + \frac{1}{3}\vec{u}_3^{"})\,\} \\
\langle \vec{v}_1^{'}\,\vec{v}_2^{'}\,\vec{v}_3^{'}|(1+P_{34})|\vec{u}_1^{'}\,\vec{u}_2^{'}\,\vec{u}_3^{'}\rangle &= \delta^3(\vec{u}_1^{'} - \vec{v}_1^{'}) \\
&\times \{\ \delta^3(\vec{u}_2^{'} + \frac{2}{3}\vec{v}_2^{'} - \frac{2}{3}\vec{v}_3^{'})\ \ \delta^3(\vec{u}_3^{'} + \frac{1}{2}\vec{v}_2^{'} + \vec{v}_3^{'}) \\
&\quad +\delta^3(\vec{u}_2^{'} + \frac{2}{3}\vec{v}_2^{'} + \frac{2}{3}\vec{v}_3^{'})\ \ \delta^3(\vec{u}_3^{'} + \frac{1}{2}\vec{v}_2^{'} - \vec{v}_3^{'})\,\}
\end{aligned} \tag{4}
$$

We can rewrite the coupled equations, $Eq.(3)$, as below coupled three-dimensional integral equations:

$$\begin{aligned}
\langle \vec{u}_1\, \vec{u}_2\, \vec{u}_3|\psi_1\rangle &= \frac{1}{E-\frac{u_1^2}{m}-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m}} \int d^3u_2^{'}\ \langle \vec{u}_1|t_s(\epsilon)|\frac{1}{2}\vec{u}_2+\vec{u}_2^{'}\rangle \\
&\times \{\ \langle \vec{u}_2+\frac{1}{2}\vec{u}_2^{'}\ \vec{u}_2^{'}\ \vec{u}_3|\psi_1\rangle \\
&\quad +\langle \vec{u}_2+\frac{1}{2}\vec{u}_2^{'}\ \frac{1}{3}\vec{u}_2^{'}+\frac{8}{9}\vec{u}_3\ \vec{u}_2^{'}-\frac{1}{3}\vec{u}_3|\psi_1\rangle \\
&\quad +\langle \vec{u}_2+\frac{1}{2}\vec{u}_2^{'}\ -\vec{u}_2^{'}-\frac{2}{3}\vec{u}_3\ \frac{1}{2}\vec{u}_2^{'}-\frac{2}{3}\vec{u}_3|\psi_2\rangle\ \} \\
\langle \vec{v}_1\, \vec{v}_2\, \vec{v}_3|\psi_2\rangle &= \frac{\frac{1}{2}}{E-\frac{v_1^2}{m}-\frac{v_2^2}{2m}-\frac{v_3^2}{m}} \int d^3v_3^{'}\,\langle \vec{v}_1|t_s(\epsilon^*)|\vec{v}_3^{'}\rangle \\
&\times \{\ 2\,\langle \vec{v}_3\ \frac{2}{3}\vec{v}_2+\frac{2}{3}\vec{v}_3^{'}\ \frac{1}{2}\vec{v}_2-\vec{v}_3^{'}|\psi_1\rangle \\
&\quad +\langle \vec{v}_3\ -\vec{v}_2\ \vec{v}_3^{'}|\psi_2\rangle\ \}
\end{aligned} \tag{5}$$

Here ϵ and ϵ^* are two-body subsystem energies and $\langle\vec{a}|t_s(\varepsilon)|\vec{b}\rangle$ is the symmetrized two-body t-matrix[10]. The so obtained Y-amplitudes fulfill the below symmetry relations, as can be seen from $Eq.(5)$.

$$\begin{aligned}
\langle \vec{u}_1\, \vec{u}_2\, \vec{u}_3|\psi_1\rangle &= \langle -\vec{u}_1\, \vec{u}_2\, \vec{u}_3|\psi_1\rangle \\
\langle \vec{v}_1\, \vec{v}_2\, \vec{v}_3|\psi_2\rangle &= \langle -\vec{v}_1\, \vec{v}_2\, \vec{v}_3|\psi_2\rangle \\
\langle \vec{v}_1\, \vec{v}_2\, \vec{v}_3|\psi_2\rangle &= \langle \vec{v}_1\, \vec{v}_2\ -\vec{v}_3|\psi_2\rangle
\end{aligned} \tag{6}$$

3. Choosing Coordinate Systems

The Y-components $|\psi_i(\vec{a}\ \vec{b}\ \vec{c})\rangle$ are given as function of Jacobi momenta vectors as solution of coupled three-dimensional integral equations, $Eq.(5)$. Since we ignore spin and isospin dependencies, for the ground state both Y-components $|\psi_i(\vec{a}\ \vec{b}\ \vec{c})\rangle$ are scalars and thus only depend on the magnitudes of Jacobi momenta and the angles between them. The first important step for an explicit calculation is the selection of independent variables, one needs six independent variables to uniquely specify the geometry of the three vectors[12]. Therefore in order to solve $Eq.(5)$ directly without introducing partial wave projection, we have to define suitable coordinate systems. For both Y-components we choose the third vector parallel to $Z-$axis, the second vector in the $X-Z$ plane and express the remaining vectors, the first as well as the integration vectors, with respect to them. With this choice of variables we can obtain the explicit representation for the Y-components $|\psi_1\rangle$ and $|\psi_2\rangle$ as [14]:

$$
\begin{aligned}
\psi_1(u_1\,u_2\,u_3\,x_1\,x_2\,x_{12}^3) = & \frac{1}{E-\frac{u_1^2}{m}-\frac{3u_2^2}{4m}-\frac{2u_3^2}{3m}} \int d^3u_2^{'}\; t_s(u_1,\tilde{\pi},\tilde{x};\epsilon) \\
& \times \{\; \psi_1(\pi_1\; u_2^{'}\; u_3\; x_{12}\; x_{13}\; x_{\pi_1 u_2'}^{u_3}) \\
& \quad +\psi_1(\pi_1\; \pi_2\; \pi_3\; x_{22}\; x_{23}\; x_{\pi_1\pi_2}^{\pi_3}) \\
& \quad +\psi_2(\pi_1\; \pi_4\; \pi_5\; x_{32}\; x_{33}\; x_{\pi_1\pi_4}^{\pi_5})\; \} \\
\psi_2(v_1\,v_2\,v_3\,X_1\,X_2\,X_{12}^3) = & \frac{\frac{1}{2}}{E-\frac{v_1^2}{m}-\frac{v_2^2}{2m}-\frac{v_3^2}{m}} \int d^3v_3^{'}\, t_s(v_1,v_3^{'},Y_{13'};\epsilon^*) \\
& \times \{\; 2\,\psi_1(v_3\; \Sigma_1\; \Sigma_2\; X_{12}\; X_{13}\; X_{v_3\Sigma_1}^{\Sigma_2}) \\
& \quad \psi_2(v_3\; v_2\; v_3'\; X_{22}\; X_{23}\; X_{v_3 v_2}^{v_3'})\; \} \qquad (7)
\end{aligned}
$$

The above coupled three-dimensional integral equations are the starting point for numerical calculations.

4. Summary

An alternative approach for four-body bound state calculations, which are based on solving the coupled Y-equations in a partial wave basis, is to work directly with momentum vector variables. We formulate the coupled Y-equations for identical particles as function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angles between them. We expect that coupled three-dimensional Y-equations can be handled in a straightforward and numerically reliable fashion.

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